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Elastic complex analysis and its applications in fracture mechanics

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Abstract

The key point in this paper is the introduction of elastic analytic functions. An elastic analytic function is a function of the form $\mathbf{u}: \mathbf{C} \rightarrow \mathbf{C}^2$ which is differentiable and satisfies equations which are analogous to the Cauchy–Riemann equations of traditional complex analysis such that the following conditions hold: first the real and imaginary part of the first complex component of $\mathbf{u}$ satisfy the Navier equation of plane elasticity, and second, the derivative along a line of the real and imaginary part of the second complex component of $\mathbf{u}$ is proportional to the applied tractions along that line. Algebraical operations have been defined on elastic analytic functions such that they constitute a commutative algebra over the real field and a module over the set of analytic functions. Next, a derivative and an integral of elastic analytic functions are introduced such that they behave in a similar way to complex differentiation and integration of analytic functions, in particular we have properties such as: the integral of an elastic analytic function around a contour is zero, a Cauchy-like integral formula and Plemelj-like formulae. These properties can be very useful in tackling problems of plane elasticity involving cracks through the boundary element method. It is also proved that path independent integrals in plane elasticity that are derived from Noether's theorem, whose integrand only depends on the position and gradient of displacements, can be written as the integral of an elastic analytic function.

Keywords: Weight functions; Plane elasticity; Fracture mechanics; Boundary elements; Path independent integrals; Complex analysis; Muskhelisavili potentials

1. Introduction

The usefulness of complex analysis in the resolution of problems governed by the Laplace equation, for example antiplane elastostatic problems, is well known. The key point is that the real and imaginary part of an analytic function in complex analysis satisfy the Laplace equation. If we are given a potential

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that satisfies the Laplace equation, we can always add its harmonic conjugate to build up an analytic function and then solve a particular problem by using all the tools of complex analysis, for example: Cauchy integral formula, analytic continuation, expansion in series, Hilbert problem, path independent integrals, etc. Furthermore, the normal derivative of the potential, which is normally associated with 'load type' boundary conditions, is equal to the derivative of the harmonic conjugate along a line (multiplied by -1 depending on the conventions). For an example of how powerful this technique can be, the reader is referred to two recent works by Atkinson and Aparicio (1994) and Aparicio and Pidcock (1996).

The techniques of complex analysis can also be applied to solve plane elastostatic problems governed by the Navier equations. This is normally done by writing the displacements and stresses as functions of complex analytic potentials. The best known ways of doing this are probably those given by MacGregor (1935), Westergaard (1939) and Muskhelishvili (1963). However, the analogy between the use of these methods in plane elastostatic problems and the use of complex analysis in antiplane elastostatic problems is apparently not obvious. For example, the displacements of a plane problem are not written as part of an 'analytic function' but a combination of analytic functions and their conjugates. This brings a gap between the mathematical armoury associated with the complex potentials and the physical interpretation of the problem. As an example, let us consider the Muskhelishvili potential $\psi(z)$. Since it is an analytic function within the region where a plane elastic state exists, we can apply to it the Cauchy integral formula or other tools of complex analysis, but what all that mean in relation to the plane elastic problem itself is rather unclear. On the other hand, if we take the complex potential whose real part is the solution of an antiplane problem in a particular region and apply to it the Cauchy integral formula, then we have a weak Green's fundamental formula that will allow us, for example, to solve the problem using the boundary element method.

The intention of this work is to construct a new complex analysis that will interact with plane elastostatic problems in a manner that is similar to the interaction between normal complex analysis and antiplane problems. For this reasons it is called *elastic complex analysis* to distinguish it from normal complex analysis. Furthermore, the adjective 'elastic' will be used throughout this paper to distinguish the mathematical operations defined within elastic complex analysis from those of normal complex analysis (i.e. elastic analytic function, elastic derivative, etc.).

In the same way that the key feature in the use of complex analysis in antiplane elastostatic problems is the concept of an analytic function, the key feature in the use of elastic complex analysis in plane elastostatic problems is the concept of an elastic analytic function. An elastic analytic function is a function $\mathbf{u}\colon \mathbf{C}\rightarrow\mathbf{C}^2$. The real and imaginary part of the first complex component of an elastic analytic function satisfy the Navier equations of plane elasticity, in other words, they work as the displacements of an elastic state. The derivative of the real and imaginary part of the second complex component along a line is proportional to the applied tractions.

The basic arithmetic operations will be defined such that the set of elastic analytic functions becomes a commutative algebra over the real field under these operations. This includes the fact that the sum and product of elastic analytic functions are also elastic analytic functions. However, the set of elastic analytic functions with the sum and product so defined will not be an integral domain unlike the set of analytic functions. A division operation will also be introduced that will allow certain elastic analytic functions to be inverted.

The product of elastic analytic functions has been introduced only as a formalism and has not been proved so far to be very useful in practical applications. A more interesting tool is the product of an elastic analytic function and an analytic function. Under this product the set of elastic analytic functions is, in general, a module over the set of analytic functions (it is, however, an algebra over the integral domain of analytic functions under a more restricted definition). Other useful tools are the elastic derivative and its inverse the elastic integral. As in normal complex analysis, the elastic integral around a domain Ω of a function which is elastic analytic in Ω is zero. This property allows us to construct path independent integrals. It will be proved that path independent integrals of the J type can be written as the elastic integral of an elastic analytic function. A formula similar to the Cauchy integral formula will also be introduced. As the Cauchy integral formula can be seen as a weak form of Green's fundamental formula, this new integral can be seen as a weak form of Somigliana's integral formula which is widely used in the resolution of elasticity problems through boundary elements. The equivalent of Plemelj formulae for this new integral formula as well as a closed form inversion will also be studied as this can be particularly useful in the resolution through boundary elements of plane elastostatic problems involving cracks in the sense that it simplifies the notation and procedure in the method employed in Aparicio and Atkinson (1997). Finally, it will be seen that elastic analytic functions can also be expanded in Taylor series and analytically continued. Indeed, they have most of the properties of normal analytic functions.

2. Elastic analytic functions

The stresses and displacements of an elastic state can be written in terms of the Muskhelishvili potentials $\phi(z)$ and $\psi(z)$ (Muskhelishvili, 1963) as

$$\frac{\sigma_{xx}+\sigma_{yy}}{2}=\phi'(z)+\overline{\phi'(z)}$$

$$\frac{\sigma_{yy}-\sigma_{xx}}{2}+i\sigma_{xy}=\bar{z}\phi''(z)+\psi'(z)$$

$$u=u_x+iu_y=\frac{1}{2G}\Big[k\phi(z)-z\overline{\phi'(z)}-\overline{\psi(z)}\Big], \tag{1}$$

where $k=(\lambda+3G)/(\lambda+G)$ is a real elastic constant greater than 1 and $G>0$ and $\lambda>0$ are the Lame constants.

We introduce the traction potential p defined by

$$p=\frac{1}{2G}\Big[\phi(z)+z\overline{\phi'(z)}+\overline{\psi(z)}\Big]. \tag{2}$$

It is called traction potential because its derivative along a line L is proportional to the applied tractions

$$\frac{\mathrm{d}p}{\mathrm{d}s}=\frac{i}{2G}\big[T_x(\mathbf{n})+iT_y(\mathbf{n})\big], \tag{3}$$

where $\mathbf{n}$ is a unit vector orthogonal to L that has been obtained by rotating 90° clockwise a unit vector tangent to L that points to the direction where the arc length s increases.

Definition. A function $\mathbf{u}$: $\mathbf{C}\rightarrow\mathbf{C}^2$ of the form $\mathbf{u}=(u,p)$ is an elastic analytic function in an open subset Ω of $\mathbf{C}$ if it is differentiable in Ω and satisfies the following equations

$$\frac{\partial u}{\partial \bar{z}}=-\frac{\partial p}{\partial \bar{z}} \tag{4}$$

$$\frac{\partial u}{\partial z}+\frac{\partial \bar{u}}{\partial \bar{z}}=k\frac{\partial \bar{p}}{\partial \bar{z}}-\frac{\partial p}{\partial z} \tag{5}$$

Eqs. (4) and (5) can be regarded as the equivalent of Cauchy–Riemann equations of complex analysis. They simply state that the Muskhelishvili potentials $\phi(z)$ and $\psi(z)$ must be analytic functions, i.e. their partial derivative with respect to $\bar{z}$ must vanish. From these equations we can obtain the conditions that u and p must satisfy in order to be the components of an elastic analytic function. Differentiating Eq. (5) with respect to $\bar{z}$ and using Eq. (4), we obtain

$$-\frac{\partial^2 p}{\partial z\partial \bar{z}}+\frac{\partial^2 \bar{u}}{\partial \bar{z}^2}=k\frac{\partial^2 \bar{p}}{\partial \bar{z}^2}-\frac{\partial^2 p}{\partial z\partial \bar{z}}. \tag{6}$$

Differentiating Eq. (5) with respect to z and using Eq. (4) we obtain

$$\frac{\partial^2 u}{\partial z^2}-\frac{\partial^2 \bar{p}}{\partial z\partial \bar{z}}=k\frac{\partial^2 \bar{p}}{\partial z\partial \bar{z}}-\frac{\partial^2 p}{\partial z^2}. \tag{7}$$

Subtracting the conjugate of Eq. (6) from Eq. (7) we find, after some minor rearrangements

$$\frac{\partial^2 p}{\partial z^2}-\frac{\partial^2 \bar{p}}{\partial z\partial \bar{z}}=0. \tag{8}$$

By proceeding in a similar way but eliminating p rather than u, we obtain

$$\frac{\partial^2 u}{\partial z^2}+k\frac{\partial^2 \bar{u}}{\partial z\partial \bar{z}}=0. \tag{9}$$

Eq. (9) is actually the Navier equation of plane elasticity as can easily be seen if it is rewritten as

$$\frac{\partial}{\partial z}\mathrm{Re}\left(\frac{\partial u}{\partial z}\right)+\frac{k-1}{2}\frac{\partial^2 \bar{u}}{\partial z\partial \bar{z}}=0$$

and we use the following relationships

$$\frac{\partial}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)$$

$$\mathrm{Re}\left(\frac{\partial u}{\partial z}\right)=\frac{1}{2}\nabla\cdot(u_x,u_y) \qquad \frac{\partial^2}{\partial z\partial \bar{z}}=\frac{1}{4}\nabla^2.$$

In order to simplify the notation in the forthcoming sections, the following matrices have been introduced

$$A_1=\frac{1}{1+k}\begin{bmatrix} k & k \\ 1 & 1\end{bmatrix},\quad A_2=\frac{1}{1+k}\begin{bmatrix} -1 & k \\ 1 & -k\end{bmatrix},\quad A_3=\frac{1}{1+k}\begin{bmatrix} 1 & 1 \\ -1 & -1\end{bmatrix}$$

These matrices have the following multiplication table

$$\begin{array}{lll} A_1^2=A_1 & A_1A_2=0 & A_1A_3=0 \\ A_2A_1=0 & A_2^2=-A_2 & A_2A_3=-A_3 \\ A_3A_1=A_3 & A_3A_2=0 & A_3^2=0 \end{array}$$

and also satisfy

$$A_1 - A_2 = I,$$

where I is the identity matrix.

If $\mathbf{u}$ is an elastic analytic function, then the following relationships hold

$$A_1 \frac{\partial \mathbf{u}}{\partial \bar{z}} = 0, \qquad A_3 \frac{\partial \mathbf{u}}{\partial \bar{z}} = 0 \tag{10}$$

$$A_3 \frac{\partial \mathbf{u}}{\partial z} - A_2 \frac{\partial \bar{\mathbf{u}}}{\partial \bar{z}} = 0. \tag{11}$$

If $\mathbf{u}_1 = (u_1, p_1)$ and $\mathbf{u}_2 = (u_2, p_2)$ are elastic analytic functions and a and b are real numbers, then it is clear that

$$a\mathbf{u}_1 + b\mathbf{u}_2 = (au_1 + bu_2, ap_1 + bp_2)$$

is also an elastic analytic function (i.e. it satisfies Eqs. (4) and (5)). The set of elastic analytic functions therefore constitutes a vector space over the real field with the usual operations of addition and multiplication by scalars.

3. The product between an elastic analytic function and an analytic function

Definition. Let Ω be an open subset of $\mathbf{C}$, $\mathbf{u}$ an elastic analytic function in Ω, f an analytic function in Ω and z_0 a complex number. The translated product between $\mathbf{u}$ and f with respect to z_0 is defined as

$$\left(\mathbf{u} \times^{z}_{z_0} f\right)(z, \bar{z}) = \left(A_1 f(z) - A_2 \overline{f(z)}\right)\mathbf{u}(z, \bar{z}) - (z - z_0) A_3 \overline{f'(z)\mathbf{u}(z, \bar{z})}. \tag{12}$$

When the translated product is written as $\times_{z_0}$, we mean that it is applied to the same variable as that used by the analytic function that follows it. In the particular case where $z_0 = 0$ or f is a constant, we will denote the product of an elastic analytic function and an analytic function as $\mathbf{u} \times f$.

The following properties can be verified using the multiplication table for A_i matrices

1. $\mathbf{u} \times_{z_0} f$ is elastic analytic in Ω.
2. If $\mathbf{u}_1$ and $\mathbf{u}_2$ are elastic analytic functions in Ω then

$$(\mathbf{u}_1 + \mathbf{u}_2) \times_{z_0} f = \mathbf{u}_1 \times_{z_0} f + \mathbf{u}_2 \times_{z_0} f$$

3. If f_1 and f_2 are analytic functions in Ω then

$$\mathbf{u} \times_{z_0} (f_1 + f_2) = \mathbf{u} \times_{z_0} f_1 + \mathbf{u} \times_{z_0} f_2.$$

4. $\mathbf{u} \times 1 = \mathbf{u}$
5. If f and g are analytic functions in Ω, then

$$\left(\mathbf{u} \times_{z_0} f\right) \times_{z_0} g = (\mathbf{u} \times_{z_0} g) \times_{z_0} f = \mathbf{u} \times_{z_0} (fg).$$

6. From the above property we have, in particular, that if f is an analytic function in Ω that does not vanish anywhere in Ω and $\mathbf{u}^* = \mathbf{u} \times_{z_0} f$ then

$$\mathbf{u} = \mathbf{u}^* \times_{z_0} \left(\frac{1}{f}\right)$$

7. If a and b are real numbers then

$$(a\mathbf{u}) \times_{z_0} (bf) = (ab)\big(\mathbf{u} \times_{z_0} f\big)$$

Properties 1 to 5 establish that the set of elastic analytic functions is a module over the set of analytic functions.

If $\mathbf{u}$ is an elastic analytic function, there is an elastic state associated with it with Muskhelishvili potentials $\phi(z)$ and $\psi(z)$. Since $\mathbf{u} \times_{z_0} f$ is also elastic analytic, we can associate to it Muskhelishvili potentials $\phi^*(z)$ and $\psi^*(z)$. Eq. (12) must therefore relate ϕ^* and ψ^* to ϕ, ψ and the analytic function f. After some manipulation, it can be proved that these relationships are

$$\phi^*(z) = f(z)\phi(z), \qquad \psi^*(z) = f(z)\psi(z) - \overline{z_0}f'(z)\phi(z).$$

Properties 5 and 6 above can be proved more easily using these relationships between Muskhelishvili potentials. Notice that when $z_0 = 0$, the mathematical meaning of this operation is simply the multiplication by f of the Muskhelishvili potentials associated with $\mathbf{u}$.

4. The elastic derivative and elastic integral of an elastic analytic function

Definition. Let Ω be an open subset of $\mathbf{C}$ and $\mathbf{u}$ an elastic analytic function in Ω. We define the elastic derivative of $\mathbf{u}$ with respect to z as

$$\frac{e\mathbf{u}}{ez} = i\left[A_1\frac{\partial \mathbf{u}}{\partial z} + A_2\frac{\partial \mathbf{u}}{\partial \bar{z}}\right]. \tag{13}$$

The elastic derivative has the following properties:

1. $e\mathbf{u}/ez$ is elastic analytic in Ω.
2. If a and b are real numbers and $\mathbf{u}_1$ and $\mathbf{u}_2$ are elastic analytic functions, then

$$\frac{e}{ez}[a\mathbf{u}_1 + b\mathbf{u}_2] = a\frac{e\mathbf{u}_1}{ez} + b\frac{e\mathbf{u}_2}{ez}.$$

   Notice that if a is a complex number and $\mathbf{u}$ is elastic analytic, $a\mathbf{u}$ is not necessarily elastic analytic.
3. If z_0 is a complex number and f is an analytic function in Ω, then

$$\frac{e}{ez}\big(\mathbf{u} \times_{z_0} f\big) = \frac{e\mathbf{u}}{ez} \times_{z_0} f + \mathbf{u} \times_{z_0} \big(if'\big).$$

4. If the elastic derivative of an elastic analytic function $\mathbf{u}$ is zero in an open set Ω, then $\mathbf{u}$ is a constant in Ω.

   Letting Eq. (13) equal to zero, we see that

$$A_1\frac{\partial \mathbf{u}}{\partial z}=0, \qquad A_2\frac{\partial \mathbf{u}}{\partial \bar{z}}=0.$$

Combining this result with Eqs. (10) and (11) we find that both $u+p$ and $kp-u$ are constants and therefore $\mathbf{u}$ is a constant in Ω. This means that the elastic derivative is a kind of total derivative rather than a partial derivative when dealing with elastic analytic functions. This is similar to complex differentiation of analytic functions.

The effect of the elastic derivative on the Muskhelishvili potentials associated with an elastic analytic function is very simple. It simply differentiates the Muskhelishvili potentials associated with $\mathbf{u}$ and multiplies them by i.

Definition. If L is a rectifiable curve in C (that is L is of bounded variation) and $\mathbf{u}$ is a function $\mathbf{u}$: $\mathbf{C}\rightarrow\mathbf{C}^2$, then the elastic integral of $\mathbf{u}$ along L is defined by

$$\int_L \mathbf{u}*\,\mathrm{d}z=-i\int_L(A_1\mathbf{u}\,\mathrm{d}z+A_2\mathbf{u}\,\mathrm{d}\bar{z}+A_3\bar{\mathbf{u}}\,\mathrm{d}z). \tag{14}$$

The elastic integral has the following properties:

1. If a and b are real numbers, then

$$\int_L(a\mathbf{u}_1+b\mathbf{u}_2)*\,\mathrm{d}z=a\int_L\mathbf{u}_1\mathrm{d}z+b\int_L\mathbf{u}_2*\mathrm{d}z$$

2. If L is a closed curve that surrounds a region Ω where $\mathbf{u}$ is elastic analytic, then

$$\oint_L \mathbf{u}*\mathrm{d}z=0.$$

To prove this, it is sufficient to show that the integrand of the above expression is an exact differential, that is

$$\frac{\partial}{\partial\bar{z}}(A_1\mathbf{u}+A_3\bar{\mathbf{u}})=\frac{\partial}{\partial z}(A_2\mathbf{u}).$$

This is straightforward to prove using the first equation in (10) and the conjugate of Eq. (11).

3. If Ω is an open subset of $\mathbf{C}$ where $\mathbf{u}$ is elastic analytic and L is an open rectifiable curve in Ω, then

$$\int_L\frac{e\mathbf{u}}{ez}*\mathrm{d}z=\Delta\mathbf{u},$$

where $\Delta\mathbf{u}$ denotes the difference between the value of $\mathbf{u}$ at the final point of L and the value of $\mathbf{u}$ at the initial point of L according to the direction of integration. To prove this assumption, we simply substitute Eq. (13) into Eq. (14). By using the multiplication table of the A_i matrices, we arrive at

$$\int_L\frac{e\mathbf{u}}{ez}*\mathrm{d}z=\int_L\left[A_1\frac{\partial\mathbf{u}}{\partial z}\,\mathrm{d}z-A_2\frac{\partial\mathbf{u}}{\partial\bar{z}}\,\mathrm{d}\bar{z}-A_3\frac{\partial\bar{\mathbf{u}}}{\partial\bar{z}}\,\mathrm{d}z\right].$$

Adding $A_1\partial\mathbf{u}/\partial\bar{z}=0$ and noting the conjugate of Eq. (11) we get

$$\int_L \frac{e\mathbf{u}}{ez} * \mathrm{d}z = \int_L \left[(A_1 - A_2) \frac{\partial \mathbf{u}}{\partial z} \, \mathrm{d}z + (A_1 - A_2) \frac{\partial \mathbf{u}}{\partial \bar{z}} \, \mathrm{d}\bar{z} \right]$$

$$= \int_L \left(\frac{\partial \mathbf{u}}{\partial z} \, \mathrm{d}z + \frac{\partial \mathbf{u}}{\partial \bar{z}} \, \mathrm{d}\bar{z} \right) = \Delta \mathbf{u}.$$

5. A Cauchy integral formula for elastic complex analysis

Theorem. *If Ω is an open subset of* $\mathbf{C}$ *with boundary L,* $\mathbf{u}$ *is an elastic analytic function in Ω and z_0 is a complex number in Ω, then*

$$\oint_L \left[\mathbf{u}(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0} \right) \right] * \mathrm{d}z = 2\pi \mathbf{u}(z_0, \overline{z_0}), \tag{15}$$

where the integral is carried out anticlockwise along L.

Proof. Let us consider a small circle L_ϵ of radius ϵ centered at z_0. Then, using the property of path independence, we have

$$\oint_L \left[\mathbf{u}(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0} \right) \right] * \mathrm{d}z = \oint_{L_\epsilon} \left[\mathbf{u}(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0} \right) \right] * \mathrm{d}z$$

$$\sim \oint_{L_\epsilon} \left[\mathbf{u}(z_0, \overline{z_0}) \times_{z_0}^{z} \left(\frac{1}{z - z_0} \right) \right] * \mathrm{d}z \quad \text{as } \epsilon \to 0^+.$$

Rewriting this integral by substituting the definition of the translated product (12) into the definition of the elastic integral (14) and using the multiplication table for the A_i matrices, we obtain

$$\oint_{L_\epsilon} \left[\mathbf{u}(z_0, \overline{z_0}) \times_{z_0}^{z} \left(\frac{1}{z - z_0} \right) \right] * \mathrm{d}z$$

$$= -\oint_{L_\epsilon} \left[\left(\frac{A_1 \, \mathrm{d}z}{z - z_0} + \frac{A_2 \, \mathrm{d}\bar{z}}{\bar{z} - \overline{z_0}} \right) \mathbf{u}(z_0, \overline{z_0}) + A_3 \overline{\mathbf{u}(z_o, \overline{z_0})} \, \mathrm{d}\left(\frac{z - z_0}{\bar{z} - \overline{z_0}} \right) \right]$$

$$= \left[A_1(2\pi) - A_2(2\pi) \right] \boldsymbol{u}(z_o, \overline{z_0}) = 2\pi \mathbf{u}(z_0, \overline{z_0}).$$

6. Plemelj like formulae for elastic complex analysis

Theorem. *Let L be a smooth line not intersecting itself and* $\mathbf{u}$: $L \rightarrow \mathbf{C}^2$ *a function of the form* $\mathbf{u} = (u, p)$ *such that: both u and p satisfy the Hölder condition on L, $u + p$ vanishes at the end points of L, is differentiable on L and its derivative satisfies the Hölder condition on L. Then the following function is elastic analytic everywhere in* $\boldsymbol{C}$ except on L.

$$\mathbf{p}(z_0, \overline{z_0}) = \frac{1}{2\pi}\int_L \mathbf{u}(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z \tag{16}$$

Proof. The function $\phi(z) = 2G(u + p)/(1 + k)$ vanishes at the end points of L, is differentiable on L and its derivative satisfies the Hölder condition on L. This means that the following functions are analytic everywhere in $\mathbf{C}$ except on L

$$\Phi(z_0) = \frac{1}{2\pi i}\int_L \frac{\phi(z)\,\mathrm{d}z}{z - z_0}, \qquad \Phi'(z_0) = \frac{1}{2\pi i}\int_L \frac{\phi'(z)\,\mathrm{d}z}{z - z_0} = \frac{1}{2\pi i}\int_L \frac{\phi(z)\,\mathrm{d}z}{(z - z_0)^2}$$

On the other hand, the function $\psi(z) = 2G\bar{p} - \bar{\phi} - \bar{z}\phi'$ also satisfies the Hölder condition on L and therefore

$$\Psi(z_0) = \int_L \frac{\psi(z)\,\mathrm{d}z}{z - z_0}$$

is also analytic in $\mathbf{C} - L$.

We introduce the following notation

$$\mathrm{R}_1(u, p) = u, \qquad \mathrm{R}_2(u, p) = p.$$

Substituting the expression of $\mathbf{u} = (u, p)$ in terms of its Muskhelishvili potentials (see Eqs. (1) and (2)) into the right-hand side of Eq. (16), we obtain

$$\frac{1}{2\pi}\mathrm{R}_1\int_L \mathbf{u}(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \ \mathrm{d}z$$

$$= \frac{1}{2G(2\pi i)}\int_L \left[\frac{k\phi}{z - z_0}\mathrm{d}z + \frac{z\overline{\phi'} + \bar{\psi}}{\bar{z} - \overline{z_0}}\,\mathrm{d}\bar{z} + \bar{\phi}\,\mathrm{d}\left(\frac{z - z_0}{\bar{z} - \overline{z_0}}\right)\right]$$

$$= \frac{1}{2G(2\pi i)}\int_L \left[\frac{k\phi}{z - z_0}\,\mathrm{d}z + \frac{z_0\overline{\phi'} + \bar{\psi}}{\bar{z} - \overline{z_0}}\,\mathrm{d}\bar{z} + \mathrm{d}\left(\bar{\phi}\frac{z - z_0}{\bar{z} - \overline{z_0}}\right)\right]$$

$$= \frac{1}{2G}\left(k\Phi(z_0) - z_0\overline{\Phi'(z_0)} - \overline{\Psi(z_0)}\right).$$

Similarly

$$\frac{1}{2\pi}\mathrm{R}_2\int_L \mathbf{u}(z,\bar{z})\times^{z}_{z_0}\left(\frac{1}{z-z_0}\right)*\mathrm{d}z=\frac{1}{2G}\left(\Phi(z_0)+z_0\overline{\Phi'(z_0)}+\overline{\Psi(z_0)}\right).$$

Thus, $\mathbf{p}(z_0,\overline{z_0})$ is the elastic analytic function associated with the Muskhelishvili potentials $\Phi(z_0)$ and $\Psi(z_0)$.

Theorem. *Let L be a simple smooth oriented curve, let z_0 be a point on L, let $\mathbf{u_j}=(u_j,p_j)$ be a function $\mathbf{u_j}\colon L\to\mathbf{C}$ satisfying the Hölder condition on L such that $u+p$ vanishes at the end points of L, is differentiable on L and its derivative satisfies the Hölder condition on L. Then if $\mathbf{U}_+(z_0,\overline{z_0})$ and $\mathbf{U}_-(z_0,\overline{z_0})$ are, respectively, the limits of*

$$\mathbf{U}(z_1,\overline{z_1})=\frac{1}{2\pi}\int_L \mathbf{u_j}(z,\bar{z})\times^{z}_{z_1}\left(\frac{1}{z-z_1}\right)*\ \mathrm{d}z$$

as z_1 approaches z_0 on the left and right of L respectively, we have

$$\mathbf{u_a}(z_0,\overline{z_0})=\frac{\mathbf{U}_+(z_0,\overline{z_0})+\mathbf{U}_-(z_0,\overline{z_0})}{2}=\frac{1}{2\pi}\mathrm{PV}\int_L \mathbf{u_j}(z,\bar{z})\times^{z}_{z_0}\left(\frac{1}{z-z_0}\right)*\ \mathrm{d}z,$$

$$\mathbf{u_j}(z_0,\overline{z_0})=\mathbf{U}_+(z_0,\overline{z_0})-\mathbf{U}_-(z_0,\overline{z_0}) \tag{17}$$

where PV *means that the integral must be interpreted in the sense of its Cauchy principal value.*

Proof. The proof can be carried out in a similar way to the previous theorem by taking into account Plemelj formulae (Muskhelishvili, 1958)

$$\frac{\Phi_+(z_0)+\Phi_-(z_0)}{2}=\frac{1}{2\pi i}\mathrm{PV}\int_L\frac{\phi(z)\,\mathrm{d}z}{z-z_0},$$

$$\Phi_+(z_0)-\Phi_-(z_0)=\phi(z_0).$$

Eq. (17) can be inverted in closed form in a similar way as for complex functions in Muskhelishvili (1958). We simply state here the result (see Appendix A for a complete proof).

$$\mathbf{u_j}(z_0,\overline{z_0})=\frac{2}{\pi}\left[\mathrm{PV}\int_L\left(\mathbf{u_a}(z,\bar{z})\times^{z}g_+(z)\right)\times^{z}_{z_0}\left(\frac{1}{z-z_0}\right)\mathrm{d}z\right]\times^{z_0}\frac{1}{g_+(z_0)}$$

$$+\mathbf{A}\times^{z_0}\frac{1}{g_+(z_0)}, \tag{18}$$

where $\mathbf{A}$ is a constant in $\mathbf{C}^2$ and $g_+(z)$ the limit of the function $g(\zeta)$

$$g(\zeta) = (\zeta - z_2)\sqrt{\frac{\zeta - z_1}{\zeta - z_2}}$$

as ζ approaches a point z on L on the left. In the above formula the branch cut is chosen along L and z_1, z_2 are the initial and final points of L, respectively.

Eq. (17) can be seen as the superposition of two cases by making $\mathbf{u_j} = \mathbf{u_{j1}} + \mathbf{u_{j2}}$, where $\mathrm{R}_2\mathbf{u_{j1}} = 0$ and $\mathrm{R}_1\mathbf{u_{j2}} = 0$ on L. The first case corresponds to a distribution of dislocations along L which produces stresses that are continuous across L. The second case corresponds to a distributed load and therefore displacements are continuous across L. These cases were already studied by Aparicio and Atkinson (1997) in the determination of weight functions for curved cracks.

Eq. (17) can be very useful in the resolution of plane elastostatic problems involving cracks. Only two of the four components of Eq. (17) need to be considered when constructing the system of equations. The choice of which complex component is more appropriate will depend on which complex component of $\mathbf{u_a}$ can be written in terms of either the variables of the problem or known quantities. For external boundaries, either u or p will be appropriate as nodal variables depending on the boundary conditions; while along cracks, where stresses are known, $u_j = u_+ - u_-$ will be a better choice for nodal variables, and the second complex component of Eq. (17) should be considered. After the system of equations has been solved, then Eq. (17) can be used again to determine either u_+ or u_- along cracks. Notice that Eq. (17) only needs to be evaluated once at each node along the crack.

This approach to solving elastostatic problems has an advantage and a disadvantage. The disadvantage is that the traction potential is the integral of the tractions and therefore has an unknown constant involved that must be included in the list of nodal variables. There is a constant for each disjointed boundary in the problem (including cracks as part of the boundaries). It is a worthy task to figure out the best ways this constant can be included into the system of equations. Since the number of these constants is probably much smaller than the number of nodal variables, its main drawback is not that it is adding more equations into the system, but rather that we must find the most appropriate equations to include them. Work dealing with these constants have already been published (see Chen, 1993 for example).

The advantage of using Eq. (17) to solve elastostatic problems numerically is that (17) is a singular integral equation with a Cauchy type kernel whose singularity is straight forward to remove, namely

$$\mathrm{PV}\int_L \mathbf{u_j}(z,\bar{z}) \times_{z_0}^{z} \left(\frac{1}{z-z_0}\right) * \,\mathrm{d}z = \mathrm{PV}\int_L \left(\mathbf{u_j}(z,\bar{z}) - \mathbf{u_j}(z_0,\overline{z_0})\right) \times_{z_0}^{z} \left(\frac{1}{z-z_0}\right) * \,\mathrm{d}z$$

$$+\mathrm{PV}\int_L \mathbf{u_j}(z_0,\overline{z_0}) \times_{z_0}^{z} \left(\frac{1}{z-z_0}\right) * \,\mathrm{d}z, \tag{19}$$

where L is for example a boundary element. The first integral on the right-hand side of Eq. (19) is non-singular, the second integral can be evaluated in closed form. If

$$\mathbf{q}_{\pm}(z_0,\overline{z_0}) = \int_L \frac{e}{ez}\left[\mathbf{u_j}(z_0,\overline{z_0}) \times_{z_0}^{z} \left(-i\ln_{\pm}(z-z_0)\right)\right] * dz$$

$$= \Delta\left\{\mathbf{u_j}(z_0,\overline{z_0}) \times_{z_0}^{z} \left[-i\ln_{\pm}(z-z_0)\right]\right\}, \tag{20}$$

where $\ln_{\pm}$ means the logarithm has the branch cut on the $\pm$ side of L, so the branch cut does not intersect L in Eq. (20) as z_0 is also on the $\pm$ side of L. Then the last integral in Eq. (19) will be equal to $(\mathbf{q}_+ + \mathbf{q}_-)/2$. Since

$$\Delta \ln_{+}(z - z_0) - \Delta \ln_{-}(z - z_0) = 2\pi i,$$

this result can be written in terms of one kind of logarithm (see Aparicio and Atkinson, 1997 for more details in a numerical implementation of this procedure).

Another alternative that has appeared in the literature is integrating Eq. (17) by parts. This is possible because

$$\frac{e}{ez}\left[\mathbf{u_j}(z,\bar{z}) \times_{z_0}^{z} \left(-i\ln_{\pm}(z-z_0)\right)\right] = \left[\frac{e\mathbf{u_j}(z,\bar{z})}{ez} \times_{z_0}^{z} \left(-i\ln_{\pm}(z-z_0)\right)\right] +$$

$$+\mathbf{u_j}(z,\bar{z}) \times_{z_0}^{z} \left(\frac{1}{z-z_0}\right).$$

Then the unknown is associated with $e\mathbf{u}/ez$ (which means that a condition of closure of the crack is needed) and the singularity is of a logarithmic type (see Chen and Cheung, 1994 and Chang and Mear, 1996 for recent works on this topic).

7. The product of two elastic analytic functions

As it was mentioned earlier, the product of two elastic analytic functions does not appear to have any useful application and we merely introduce it here as a formalism. The aim is to define a product between two elastic analytic functions such that this product and the addition of elastic analytic functions introduced previously constitute a commutative ring (this includes the fact that the product of two elastic analytic functions is also an elastic analytic function); and the elastic derivative of the product of two elastic analytic functions operates in a similar way to the derivative of the product of two analytic functions.

Let ϕ and ψ be two analytic functions. In order to simplify the notation, we define

$$\left[\phi(z), \psi(z)\right] = \frac{1}{2G}\left(k\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}, \phi(z) + z\overline{\phi'(z)} + \overline{\psi(z)}\right).$$

In other words, $[\phi, \psi]$ is the elastic analytic function associated with the Muskhelishvili potentials ϕ and ψ. We also have as a consequence of this that

$$\mathbf{u} = 2G\left[\mathrm{R}_1(A_3\mathbf{u}), \mathrm{R}_1\left(A_2\bar{\mathbf{u}} - \bar{z}A_3\left(\frac{\partial \mathbf{u}}{\partial z}\right)\right)\right].$$

Definition. If $\mathbf{u}_1 = [\phi_1, \psi_1]$ and $\mathbf{u}_2 = [\phi_2, \psi_2]$ are two elastic analytic functions defined on the same open subset of the complex plane and z_0 is a complex number, then the translated product of these functions is defined as

$$\mathbf{u}_1 \times_{z_0} \mathbf{u}_2 = \left[\phi_1\phi_2, \left(\psi_1 + \overline{z_0}\phi_1'\right)\left(\psi_2 + \overline{z_0}\phi_2'\right) - \overline{z_0}\left(\phi_1\phi_2\right)'\right]. \tag{21}$$

The product of two elastic analytic functions satisfies the following properties:

1. The product and addition defined for elastic analytic functions constitute a commutative ring. The

neutral element for the addition is $\mathbf{0} = (0, 0)$ and the neutral element for the product is $\mathbf{1} = (k-1, 2)/(2G) = [1, 1]$.
2. $e(\mathbf{u_1} \times_{z_0} \mathbf{u_2})/ez = (e\mathbf{u_1}/ez) \times_{z_0} \mathbf{u_2} + \mathbf{u_1} \times_{z_0} (e\mathbf{u_2}/ez)$.
3. $(\mathbf{u_1} \times_{z_0} \mathbf{u_2}) \times_{z_0} f = \mathbf{u_1} \times_{z_0} (\mathbf{u_2} \times_{z_0} f) = (\mathbf{u_1} \times_{z_0} f) \times_{z_0} \mathbf{u_2}$.

This last property proves that when the translated product $\times$ between two elastic analytic functions and between an elastic analytic function and an analytic function is applied with the same complex number z_0, the set of elastic analytic functions forms an algebra over the integral domain of analytic functions.

Definition. The transpose of an elastic analytic function $\mathbf{u} = [\phi, \psi]$ is defined as

$$\mathbf{u}^{\mathrm{T}} = \left[\psi + \overline{z_0}\phi',\ \phi - \overline{z_0}\big(\psi + \overline{z_0}\phi'\big)'\right].$$

Definition. The determinant of an elastic analytic function $\mathbf{u} = [\phi, \psi]$ is defined as

$$\det(\mathbf{u}) = \phi\big(\psi + \overline{z_0}\phi'\big).$$

Definition. The inverse of an elastic analytic function $\mathbf{u} = [\phi, \psi]$ is defined as

$$\mathbf{u}^{-1} = \left[\frac{1}{\phi}, \frac{1}{\psi + \overline{z_0}\phi'} + \overline{z_0}\phi'/\phi^2\right].$$

The following properties can easily be verified

1. The transpose of an elastic analytic function is also an elastic analytic function. The determinant of an elastic analytic function is an analytic function.
2. $\mathbf{u} \times_{z_0} \mathbf{u}^{\mathrm{T}} = 1 \times_{z_0} \det(\mathbf{u})$. From this property, we can conclude that if an elastic analytic function in Ω has a determinant that does not vanish anywhere in Ω, then it has an inverse given by

$$\mathbf{u}^{-1} = \mathbf{u}^{\mathrm{T}} \times_{z_0} \left(\frac{1}{\det(\mathbf{u})}\right)$$

which is also an elastic analytic function in Ω.
3. $\det(\mathbf{u}^{\mathrm{T}}) = \det(\mathbf{u})$.
4. $\det(\mathbf{u}^{-1}) = 1/\det(\mathbf{u})$.
5. $(\mathbf{u}^{\mathrm{T}})^{\mathrm{T}} = \mathbf{u}$.
6. $(\mathbf{u} \times_{z_0} f)^{\mathrm{T}} = \mathbf{u}^{\mathrm{T}} \times_{z_0} f$.
7. $(\mathbf{u} \times_{z_0} f)^{-1} = \mathbf{u}^{-1} \times_{z_0} (1/f)$.
8. $(\mathbf{u_1} \times_{z_0} \mathbf{u_2})^{\mathrm{T}} = \mathbf{u_1}^{\mathrm{T}} \times_{z_0} \mathbf{u_2}^{\mathrm{T}}$.
9. $\det(\mathbf{u_1} \times_{z_0} \mathbf{u_2}) = \det(\mathbf{u_1})\det(\mathbf{u_2})$.
10. $(\mathbf{u_1} \times_{z_0} \mathbf{u_2})^{-1} = \mathbf{u}_1^{-1} \times_{z_0} \mathbf{u_2}^{-1}$.
11. If $\mathbf{u} = \mathbf{u}^{\mathrm{T}}$, then $\mathbf{u}$ can be written as $\mathbf{u} = \mathbf{1} \times_{z_0} f$ where f is an analytic function. In this case it is said

that **u** is symmetrical. A typical example of a symmetrical elastic analytic function is

$$\mathbf{u} \times_{z_0} \mathbf{u}^{\mathrm{T}}.$$

12.

$$\frac{e\mathbf{u}^{\mathrm{T}}}{ez} = \left(\frac{e\mathbf{u}}{ez}\right)^{\mathrm{T}}.$$

13. $e\mathbf{u}^{-1}/ez = (e\mathbf{u}/ez) \times_{z_0} (-\mathbf{u}^{-2})$.

The set of elastic analytic functions in a certain open subset Ω of the complex plane can also be written as the direct sum of two principal ideals I_ϕ and I_ψ generated by

$$\mathbf{1}_\phi = [1, 0], \qquad \mathbf{1}_\psi = [0, 1],$$

respectively. We conclude this section by pointing out that the product of elastic analytic functions is not an integral domain. If $\mathbf{u_1} \times_{z_0} \mathbf{u_2} = \mathbf{0}$, it does not necessarily mean that either $\mathbf{u_1} = \mathbf{0}$ or $\mathbf{u_2} = \mathbf{0}$. However, if $\mathbf{u_1}$ has a non-vanishing determinant then it can be concluded from $\mathbf{u}_1 \times_{z_0} \mathbf{u}_2 = 0$ that $\mathbf{u}_2 = \mathbf{0}$.

8. Taylor expansions of elastic analytic functions

The property that elastic analytic functions can be expanded in Taylor series is a direct consequence of the fact that Muskhelishvili potentials associated with it can be expanded in Taylor series. Let us assume that an elastic analytic function **u** has an expansion

$$\mathbf{u}(z,\bar{z}) = \sum_{n=0}^{\infty} \boldsymbol{a}_n \times_{z_0}^{z} (z - z_0)^n. \tag{22}$$

Taking the elastic derivative n times and evaluating the resulting expression for $z = z_0$, we obtain

$$\frac{\mathrm{e}^n\mathbf{u}}{ez^n}(z_0, \overline{z_0}) = \mathbf{a}_n \times (n!i^n).$$

Substituting the above expression into Eq. (22) we obtain the Taylor expansion of an elastic analytic function

$$\mathbf{u}(z, \bar{z}) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\mathrm{e}^n\mathbf{u}}{ez^n}(z_0, \overline{z_0}) \times_{z_0}^{z} \left(\frac{z - z_0}{i}\right)^n\right].$$

9. Path independent integrals

The subject of path independent integrals in the theory of elasticity has interested the scientific community working in the area since the introduction of the J (or F_l) integral by Eshelby (1956) and Rice (1968). The main implication of this path independent integral lies in its interpretation as the 'force on a defect' and its link with the energy release rate and the stress intensity factor in cracks. Günther

(1962) and Knowles and Sternberg (1972) derived two more path independent integrals by applying a restricted version of Noether's theorem (Noether, 1918) to the equations of elasticity (see also Eshelby, 1975). Edelen (1981) correctly suggested that more path independent integrals could be derived from variational principles in elastostatics. Olver, 1984a, 1984b proved that in three dimensions there is a finite number of non-trivial conservation laws depending on the position, displacement and its gradient, while in two dimensions there is an infinite number of families of independent symmetries and conservation laws. Some of these path independent integrals were already derived by Tsamasphyros and Theocaris (1982) using the property that analytic compositions of Muskhelishvili potentials are analytic functions. For other works on path independent integrals, the reader is referred to Ioakimidis and Anastasselou (1993), Dong (1994) and Aparicio and Atkinson (1997).

In the case of a two-dimensional elastostatic medium which is also linear homogeneous and isotropic, Olver (1984a) and Olver (1984b) proved that all path independent integrals that depend only on the derivatives of the displacements are of the form

$$A^* = 2G(\lambda + 2G)\xi\frac{\partial B}{\partial \eta} + (\lambda + G)i\bar{B} + C, \tag{23}$$

where A^* is the complex density of the path independent integral, i.e. the integral is of the form

$$\oint_{\partial\Omega}\left[\mathrm{Re}(A^*)n_x + \mathrm{Im}(A^*)n_y\right]\,\mathrm{d}s = \mathrm{Re}\oint_{\partial\Omega} iA^*\,\mathrm{d}\bar{z} = 0;$$

ξ is given by

$$\xi = 2\frac{\partial u}{\partial \bar{z}} = -\frac{1}{G}\left(z\overline{\phi''(z)} + \overline{\psi'(z)}\right),$$

η is given by

$$\eta = \frac{2i(\lambda + 2G)\overline{\phi'(z)}}{\lambda + G},$$

and B and C are analytic functions of η.

In order to write Eq. (23) in the same notation that has been used throughout this paper, we introduce the functions

$$F_1(\phi') = -(\lambda + G)\overline{B(\eta)},\ F_2(\phi') = -i\overline{C(\eta)}, \quad A = iA^*.$$

By substituting the above expression into Eq. (23) we obtain

$$A = \left(z\overline{\phi''(z)} + \overline{\psi'(z)}\right)\overline{F_1'(\phi')} + F_1(\phi') + \overline{F_2(\phi')}. \tag{24}$$

It can easily be verified that $\partial A/\partial z$ is a real function and therefore, from Green's theorem in complex form

$$\mathrm{Re}\oint_{\partial\Omega} A\,\mathrm{d}\bar{z} = 2\mathrm{Im}\int_{\Omega}\frac{\partial A}{\partial z}\,\mathrm{d}\Omega$$

we obtain

$$\mathrm{Re}\oint_{\partial\Omega} A\, \mathrm{d}\bar{z} = 0. \tag{25}$$

In the more general case where the symmetry conservation law is also allowed to depend on the position, Eq. (24) becomes

$$A = \left(z\overline{\phi''(z)} + \overline{\psi'(z)}\right)\overline{\frac{\partial F_1(z,\phi')}{\partial\phi'}} + F_1(z,\phi') + \overline{F_2(z,\phi')} + z\overline{\frac{\partial F_1(z,\phi')}{\partial z}}. \tag{26}$$

Here, both $F_1(z,\phi')$ and $F_2(z,\phi')$ are analytic functions in each of their arguments.

It will be proved in the remaining part of this section that path independent integrals of the form Eq. (25) with a complex density given by Eq. (26) can be written as the real or imaginary part of

$$\mathrm{R}_2\oint_{\partial\Omega} \mathbf{u} * \mathrm{d}z = 0$$

where $\mathbf{u}$ is an elastic analytic function in Ω.

Since $F_1(z,\phi')$ and $F_2(z,\phi')$ are arbitrary analytic functions, without loss of generality, Eq. (26) can be written as

$$A = \left(z\overline{\phi''(z)} + \overline{\psi'(z)}\right)\overline{\frac{\partial F_1(z,\phi')}{\partial\phi'}} + F_1(z,\phi') + \overline{F_1(z,\phi')} + \overline{F_2(z,\phi')} + z\overline{\frac{\partial F_1(z,\phi')}{\partial z}}. \tag{27}$$

We introduce the conjugate expression of Eq. (27)

$$A^\dagger = -\left(z\overline{\phi''(z)} + \overline{\psi'(z)}\right)\overline{\frac{\partial F_1(z,\phi')}{\partial\phi'}} + F_1(z,\phi') + \overline{F_1(z,\phi')} - \overline{F_2(z,\phi')} - z\overline{\frac{\partial F_1(z,\phi')}{\partial z}}. \tag{28}$$

It can easily be verified that $iA^\dagger$ is also the complex density of a path independent integral of the form (25). In other words, we can construct a complex path independent integral

$$\mathrm{Re}\oint_{\partial\Omega} A\, \mathrm{d}\bar{z} + i\mathrm{Re}\oint_{\partial\Omega} iA^\dagger\, \mathrm{d}\bar{z} = 0.$$

Through elementary manipulations, the above integral can also be written as

$$\oint_{\partial\Omega}\left[\left(\frac{\bar{A} + \overline{A^\dagger}}{2}\right)\mathrm{d}z + \left(\frac{A - A^\dagger}{2}\right)\mathrm{d}\bar{z}\right] = 0.$$

Substituting Eqs. (27) and (28) into the above expression we obtain

$$\oint_{\partial\Omega}\left\{\left(F_1 + \overline{F_1}\right)\mathrm{d}z + \left[\left(z\overline{\phi''} + \overline{\psi'}\right)\overline{\frac{\partial F_1}{\partial\phi'}} + \overline{F_2} + z\overline{\frac{\partial F_1}{\partial z}}\right]\mathrm{d}\bar{z}\right\} = 0. \tag{29}$$

Let $\mathbf{u}$ be the elastic analytic function in Ω defined by

$$\mathbf{u} = 2G\left[iF_1, i\psi'\frac{\partial F_1}{\partial\phi'} + iF_2\right]. \tag{30}$$

Substituting the above expression into Eq. (14) and comparing the result with Eq. (29), we conclude that

$$\mathrm{Re}\oint_{\partial\Omega} A\,\mathrm{d}\bar{z} + i\mathrm{Re}\oint_{\partial\Omega} iA^{\dagger}\,\mathrm{d}\bar{z} = R_2\oint_{\partial\Omega} \mathbf{u} * \mathrm{d}z = 0. \tag{31}$$

An immediate implication from this result is that conservation laws with a complex density of the form (27) can be seen as the equilibrium condition of a 'transformed' elastic state. If $\mathbf{u}$ is an elastic analytic function in Ω and z_0 a complex number in Ω, then we can define another elastic analytic function $\mathbf{p}$ in Ω as

$$\mathbf{p}(z,\bar{z}) = \int_{z_0}^{z} u(\zeta,\bar{\zeta}) * \mathrm{d}\zeta$$

where the integral is carried along any path entirely contained in Ω which goes from z_o to z. If $\mathbf{u}$ is associated with a particular conservation law as described above, then Eq. (31) can be seen as the equilibrium of forces in a closed subset of Ω associated with the elastic analytic function $\mathbf{p}$.

10. The asymptotic behaviour of an elastic analytic function near a crack tip

Let us consider a crack with tip at z_0 and whose tangent at its tip makes an angle α with respect to the horizontal ($\alpha = 0$ means the crack goes to the right of z_0). The asymptotic behaviour of an elastic analytic function near the crack tip is given by

$$\mathbf{u}(z,\bar{z}) \sim \mathbf{u}(z_0,\overline{z_0}) + \frac{1}{\sqrt{2\pi}}\mathbf{k} \times_{z_0}^{z} \sqrt{z - z_0} \tag{32}$$

as $z \rightarrow z_o$. Here $\mathbf{k}$ is a constant function given by

$$\mathbf{k} = \left[-(K_{\mathrm{II}} + iK_{\mathrm{I}})\mathrm{e}^{i\alpha/2}, \left(\frac{3}{2}K_{\mathrm{II}} - \frac{i}{2}K_{\mathrm{I}}\right)\mathrm{e}^{-3i\alpha/2} \right]$$

where K_{I} and K_{II} are the mode I and II stress intensity factors, respectively. It can be verified that Eq. (32) leads to the well known asymptotic behaviour of an in-plane elastic state near a crack tip (Freund, 1990). The asymptotic behaviour of the Muskhelishvili potentials of an in-plane elastic state near the crack tip of an inclined crack can be seen explicitly written in Aparicio and Atkinson (1997).

On the other hand, the J_1 and J_2 path independent integrals can be written in the form (27) and (28), respectively by making

$$F_1(z,\phi') = -\left(\frac{1+k}{4G}\right) i(\phi')^2, \qquad F_2(z,\phi') = 0.$$

Substituting the above expressions into Eq. (30), we obtain

$$\mathbf{u}^* = \frac{1+k}{2}\left[(\phi')^2, 2\phi'\psi'\right] = (1+k)G\frac{e\mathbf{u}}{ez}(\mathbf{1} + \mathbf{1}_{\psi}) \times \left(R_2 A_3 \frac{e\mathbf{u}}{ez}\right), \tag{33}$$

where $\mathbf{u} = [\phi, \psi]$, and therefore

$$J_1 + iJ_2 = R_2\int_L \mathbf{u} * \mathrm{d}z = (1+k)GR_2\int_L \frac{e\mathbf{u}}{ez}(\mathbf{1} + \mathbf{1}_{\psi}) \times \left(R_2 A_3 \frac{e\mathbf{u}}{ez}\right) * \mathrm{d}z.$$

From Eq. (32) we obtain that

$$R_2 A_3 \frac{e\mathbf{u}}{ez} = \frac{i(K_{\mathrm{II}} + iK_{\mathrm{I}})\mathrm{e}^{i\alpha/2}}{4G\sqrt{2\pi}\sqrt{z - z_0}}, \tag{34}$$

and

$$\frac{e\mathbf{u}}{ez}(\mathbf{1} + \mathbf{1}_\psi) = \frac{1}{2\sqrt{2\pi}}\left[-\frac{i(K_{\mathrm{II}} + iK_{\mathrm{I}})\mathrm{e}^{i\alpha/2}}{\sqrt{z - z_0}}, \frac{i(3K_{\mathrm{II}} - iK_{\mathrm{I}})\mathrm{e}^{-3i\alpha/2}}{\sqrt{z - z_0}} - \frac{i\overline{z_0}(K_{\mathrm{II}} + iK_{\mathrm{I}})\mathrm{e}^{i\alpha/2}}{\sqrt{(z - z_0)^3}} \right]. \tag{35}$$

Substituting Eqs. (34) and (35) into Eq. (33) we obtain

$$\mathbf{u}^* = -\frac{1 + k}{16\pi}\left[\left(K_{\mathrm{I}}^2 - K_{\mathrm{II}}^2 - 2iK_{\mathrm{I}}K_{\mathrm{II}}\right)\mathrm{e}^{i\alpha}, \left(K_{\mathrm{I}}^2 + 3K_{\mathrm{II}}^2 + 2iK_{\mathrm{I}}K_{\mathrm{II}}\right)\mathrm{e}^{-i\alpha}\right] \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right).$$

Integrating along any contour $\partial\Omega$ that surrounds z_0 and applying the Cauchy integral formula (15), we find the values of the J_i integral around a crack tip

$$J_1 + iJ_2 = R_2 \oint_{\partial\Omega} u^* * \mathrm{d}z = -\frac{1 + k}{8G}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2 - 2iK_{\mathrm{I}}K_{\mathrm{II}}\right)\mathrm{e}^{i\alpha}.$$

This formulation is consistent with the equation $J_i = G_{ij}m_j$ where

$$\mathbf{m} = (-\cos\alpha, -\sin\alpha)$$

is a unit vector tangent to the crack at its tip and pointing to the direction of crack advance and G_{ij} is the energy release rate tensor introduced by Atkinson and Aparicio (1999) which is defined by

$$G_{11} = G_{22} = \frac{1 + k}{8G}\left(K_{\mathrm{I}}^2 + K_{\mathrm{II}}^2\right)$$

$$G_{12} = -G_{21} = \frac{1 + k}{8G}(2K_{\mathrm{I}}K_{\mathrm{II}}).$$

In the particular case where the crack is horizontal starting at its tip towards the left ($\alpha = \pi$ and $\mathbf{m} = (1, 0)$) and under mode I loading, we obtain the well-known formula for the value of J_1 around the crack tip.

$$J_1 = \frac{(1 + k)K_{\mathrm{I}}^2}{8G}$$

11. A fundamental solution for the determination of the stress intensity factor

The J integral described in the previous section has become very useful for the determination of the stress intensity factor of a straight stress free crack in a symmetrical elasticity problem, but it is of little help in the more general situation of a curved crack under mixed mode loading. Aparicio and Atkinson

(1997) devised a fundamental solution that allows us to pick up the chosen stress intensity factor of a curved internal or surface breaking crack under mixed mode loading when it is applied in conjunction with Betti's reciprocal theorem. The fundamental solution will also allow us to determine the weight function for a particular geometry by superimposing a simple crack solution which, in the most general case, will need to be solved numerically. It will be shown in this section that, with the notation of elastic complex analysis, the fundamental solution described in Aparicio and Atkinson (1997) has a simple and natural form. It will also be shown that the calculations required to prove the properties of the fundamental solution can be carried out easier with the notation of elastic complex analysis than with the classical notation of the theory of elasticity.

The fundamental solution described in Aparicio and Atkinson (1997) has the following form in elastic complex analysis

$$\mathbf{u}_1 = \mathbf{A} \times_{z_0}^{z} \sqrt{\frac{z - z_1}{z - z_0}} - (R_1\mathbf{A}, 0), \tag{36}$$

where $\mathbf{A}$ is a constant in $\mathbf{C}^2$ given by

$$\mathbf{A} = \left[-\overline{A}e^{3i\alpha/2}, \frac{B}{2}e^{-i\alpha/2} \right],$$

α is the inclination of the tangent to the crack at its tip measured anticlockwise from the positive x axis to the part of the tangent that points towards the crack; z_0 is the complex form of the point at the crack tip; z_1 is the complex form of the point at the other extreme of the crack (where the stress intensity factor is not of interest) or a point outside the body if the crack is a surface breaking crack; A and B are given by

$$A = \frac{2Gi}{\sqrt{2\pi}(1+k)} \frac{J_1 + iJ_2}{\sqrt{\overline{z_0} - \overline{z_1}}}$$

$$B = \frac{2Gi}{\sqrt{2\pi}(1+k)} \frac{3J_1 + iJ_2}{\sqrt{z_0 - z_1}}$$

and J_1, J_2 are arbitrary numbers. The square root in Eq. (36) is defined such that the branch cut runs along the crack.

The objective is the evaluation of the integral

$$I = \mathrm{Re}\int_{\partial\Omega} \left(\overline{u_2}T_1 - u_1\overline{T_2}\right) \mathrm{d}s$$

where the subindex 1 denotes the elastic state given in (36), the subindex 2 denotes the current elastic state acting upon the medium, $u = u_x + iu_y$ is the complex displacement and $T = T_x + iT_y$ is the complex traction. The integral is carried out anticlockwise around the crack tip.

Since the above integral is path independent, we only need to calculate its value around a small circle that surrounds the crack tip. In this situation, the current elastic state acquires the asymptotic behaviour given by Eq. (32)

$$\mathbf{u}_2(z, \bar{z}) \sim \mathbf{u}_2(z_0, \overline{z_0}) + \frac{\boldsymbol{k}}{\sqrt{2\boldsymbol{\pi}}} \times_{z_0}^{z} \sqrt{z - z_0}.$$

Using Eq. (3) and integrating by parts (and using the fact that $p_1 = 0$ on the crack faces as we approach the crack tip), Betti's reciprocal theorem can be written as

$$I = 2G\mathrm{Im}\int_{\partial\Omega}\left(u_1\frac{\mathrm{d}\overline{p_2}}{\mathrm{d}s} - p_1\frac{\mathrm{d}\overline{u_2}}{\mathrm{d}s}\right)\mathrm{d}s \tag{37}$$

The derivatives involved in the above integral can be written in terms of the elastic derivative

$$\left(\frac{\mathrm{d}\overline{u_2}}{\mathrm{d}s}, \frac{\mathrm{d}\overline{p_2}}{\mathrm{d}s}\right) = \frac{\mathrm{d}\overline{\mathbf{u}_2}}{\mathrm{d}s} = \overline{\frac{e\mathbf{u}_2}{ez} * \frac{\mathrm{d}z}{\mathrm{d}s}},$$

where $\mathrm{d}z/\mathrm{d}s = i\mathrm{e}^{i\theta}((r,\ \theta)$ is defined such that $z = z_0 + r\mathrm{e}^{i\theta})$ and, from the properties of the elastic derivative,

$$\frac{e\mathbf{u}_2}{ez} = \frac{\mathbf{k}}{\sqrt{2\boldsymbol{\pi}}} \times_{z_0}^{z} \left(\frac{i}{2\sqrt{z - z_0}}\right).$$

Using Eq. (12) and $z = z_0 + r\mathrm{e}^{i\theta}$, we obtain

$$\frac{e\mathbf{u}_2}{ez} \sim \frac{i}{\sqrt{2\pi r}}\left(\frac{A_1\mathbf{k}}{2}\mathrm{e}^{-i\theta/2} + \frac{A_2\mathbf{k}}{2}\mathrm{e}^{i\theta/2} - \frac{A_3\bar{\mathbf{k}}}{4}\mathrm{e}^{5i\theta/2}\right)$$

as $r \rightarrow 0^+$. Substituting the above equation into

$$\frac{d\mathbf{u}_2}{\mathrm{d}s} = \frac{e\mathbf{u}_2}{ez} * \frac{\mathrm{d}z}{\mathrm{d}s} = -i\left(A_1\frac{e\mathbf{u}_2}{ez}\frac{\mathrm{d}z}{\mathrm{d}s} + A_2\frac{e\mathbf{u}_2}{ez}\frac{\overline{\mathrm{d}z}}{\mathrm{d}s} + A_3\frac{\overline{e\mathbf{u}_2}}{ez}\frac{\mathrm{d}z}{\mathrm{d}s}\right)$$

and taking the complex conjugate, we obtain

$$\frac{\overline{\mathrm{d}\mathbf{u}_2}}{\mathrm{d}s} \sim \frac{i}{\sqrt{2\pi r}}\left(-\frac{A_1\bar{\mathbf{k}}}{2}\mathrm{e}^{-i\theta/2} - \frac{A_2\bar{\mathbf{k}}}{2}\mathrm{e}^{i\theta/2} + \frac{3A_3\mathbf{k}}{4}\mathrm{e}^{-3i\theta/2}\right) \tag{38}$$

On the other hand, developing the translated product in Eq. (36) (which is given in Eq. (12)) we obtain

$$\mathbf{u}_1 = \frac{1}{\sqrt{r}}\left(A_1\boldsymbol{A}\sqrt{z_0 - z_1}\mathrm{e}^{-i\theta/2} - A_2\mathbf{A}\sqrt{\overline{z_0} - \overline{z_1}}\mathrm{e}^{i\theta/2} + \frac{A_3\overline{\mathbf{A}}}{2}\sqrt{\overline{z_0} - \overline{z_1}}\mathrm{e}^{5i\theta/2}\right).$$

The integration can be carried out easily by taking into account that

$$\int_{\alpha}^{\alpha+2\pi}\mathrm{e}^{in\theta}\,\mathrm{d}\theta = 2\pi\delta_{0n}$$

where n is an integer. From this, we obtain

$$2G\int_{\partial\Omega}u_1\frac{\mathrm{d}\overline{p_2}}{\mathrm{d}s}\,\mathrm{d}s = -\frac{1}{4(1+k)}\left[k(J_1 - iJ_2)(3K_{\mathrm{II}} - iK_{\mathrm{I}}) + (3J_1 - iJ_2)(K_{\mathrm{II}} - iK_{\mathrm{I}})\right]$$

$$2G\int_{\partial\Omega}p_1\frac{\mathrm{d}\overline{u_2}}{\mathrm{d}s}\,\mathrm{d}s = \frac{1}{4(1+k)}\left[(J_1 - iJ_2)(3K_{\mathrm{II}} - iK_{\mathrm{I}}) + k(3J_1 - iJ_2)(K_{\mathrm{II}} - iK_{\mathrm{I}})\right]$$

and, therefore,

$$I = \mathrm{Re}\int_{\partial\Omega} \left(\overline{u_2}T_1 - u_1\overline{T_2}\right)\,\mathrm{d}s = J_1K_{\mathrm{I}} + J_2K_{\mathrm{II}}.$$

The fundamental solution given by Eq. (36) can help us to calculate the weight function for any crack geometry and boundary conditions. The weight function can be written as $\mathbf{h} = \mathbf{v} + \mathbf{u}_1$, where $\mathbf{v}$ is a simple crack solution with boundary conditions given. Those boundary conditions are determined such that $\mathbf{h}$ is stress free along the crack and on the regions of the external boundary where displacements are unknown in the original problem ($\mathbf{u}_2$), $\mathbf{h}$ also has clamped displacements on the remaining regions of the external boundary (where tractions are unknown in the original problem). (see Aparicio and Atkinson, 1997 for more details about this procedure).

12. Conclusions

We have defined elastic analytic functions, that is, functions of the kind $\mathbf{u}\colon \mathbf{C}\rightarrow\mathbf{C}^2$ that obey Eqs. (4) and (5). We have established that the real and imaginary part of the first complex component satisfy the Navier equation of plane elasticity (homogeneous, linear and isotropic) and the real and imaginary part of the second complex component behave like a traction potential (its derivative along a line is proportional to the applied tractions on it) and satisfy the Navier equation for plane strain with a Poisson coefficient equal to unity.

Algebraical operations have been defined such that the set of elastic analytic functions constitutes a commutative algebra over the real field and a module over the set of analytic functions. Additionally, a derivative and an integral have been defined such that they will interact with elastic analytic functions like complex differentiation and integration do with normal analytic functions. We have in particular that the elastic integral around a contour, enclosing a region where the integrand is elastic analytic, vanishes. We also have derived a formula similar to the Cauchy integral formula and Plemelj formulae for elastic analytic functions. Finally we have established that path independent integrals of the *J*-type (that is, path independent integrals that can be derived from Noether's theorem whose integrand only depends on the position and derivatives of the displacements) can be written as the elastic integral of an elastic analytic function.

Elastic complex analysis can provide additional tools in the resolution of two dimensional elastostatic problems in linearly homogeneous and isotropic media. It can be particularly useful in two dimensional problems involving cracks since the Cauchy integral formula for elastic analytic functions is a weak version of Somigliana integral formula with a Cauchy type singularity and has the equivalent of Plemelj formulae. The method used in Aparicio and Atkinson to determine numerically a weight function for a curved crack is based on this. Elastic complex analysis provides an easier and natural notation as well as a better understanding of the procedure for the method described in the above paper. Elastic complex analysis also provides a better understanding of the theory behind path independent integrals in two dimensional elasticity.

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Appendix A. Derivation of the inverse formula (18)

Let L be an open smooth oriented curve not intersecting itself that goes from z_1 to z_2, let g be a complex function defined as

$$g(z) = (z - z_2)\sqrt{\frac{z - z_1}{z - z_2}},$$

where the branch cut of the square root has been chosen to be along L. Let $\mathbf{u}_j = (u_j, p_j)$ be a function $\mathbf{u}_j$: $L \rightarrow \mathbf{C}$ that satisfies the Hölder condition on L such that $u + p$ vanishes at the end points of L, is differentiable on L and its derivative satisfies the Hölder condition on L. It is the aim of this section to find a closed form inversion formula for the integral equation

$$\mathbf{u}_{\mathrm{a}}(z_0, \overline{z_0}) = \frac{1}{2\pi}\mathrm{PV}\int_L \mathbf{u}_j(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z.$$

Let us define the function

$$\mathbf{q}(z_0, \overline{z_0}) = u(z_0, \overline{z_0}) \times^{z_0} g(z_0) = \left[\frac{1}{2\pi}\int_L \mathbf{u}_j(z, \bar{z}) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z\right] \times^{z_0} g(z_0)$$

$\mathbf{q}$ is a function which is elastic analytic in $\mathbf{C} - L$. We can therefore apply to it formula (15) in a region surrounded by a contour L_1 that encloses L in the clockwise direction and a circular contour L_ϵ centered at some point near L and a large radius ϵ. The result is

$$\int_L \mathbf{q}_+ \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z - \int_L \mathbf{q} - \times_{z0}^{z}\left(\frac{1}{z - z_0}\right) * \mathrm{d}z$$

$$+\oint_{L_\epsilon} \boldsymbol{q} \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z = 2\pi\mathbf{q}(z_0, \overline{z_0}) \tag{A1}$$

The integral on the right-hand side of Eq. (16) has a Cauchy type singularity, and hence, $\mathbf{u}(z, \bar{z})$ is of order $O(1/z)$ as $z \rightarrow \infty$. Since $g(z)$ is of order $O(z)$ as $z \rightarrow \infty$, we conclude that $(\mathbf{q}(z, \bar{z})$ is of order $O(1)$ as $z \rightarrow \infty$. This means that the last integral in the above equation tends to a constant $\mathbf{A}$ as $\epsilon \rightarrow \infty$. By noting that

$$\mathbf{q}_+ - \mathbf{q}_- = \mathbf{u}_+ \times g_+ - u_- \times g_- = (\mathbf{u}_+ + \mathbf{u}_-) \times g_+ = 2\mathbf{u}_a \times g_+$$

since $g_- = g_+$, we obtain from (A1)

$$\int_L (2\mathbf{u}_a \times g_+) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \ \mathrm{d}z + \mathbf{A} = 2\pi(\mathbf{u} \times g)(z_0, \overline{z_0}).$$

Using Eq. (17), we establish that

$$2\pi\frac{\mathbf{u}_+ \times g_+ + \mathbf{u}_- \times g_-}{2} = \mathrm{PV}\int_L (2\mathbf{u}_{\mathrm{a}} \times g_+) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z + \mathbf{A},$$

and since $\mathbf{u}_+ \times g_+ + \mathbf{u}_- \times g_- = \mathbf{u}_j \times g_+$ we obtain

$$(\mathbf{u}_j \times g_+)(z_0, \overline{z_0}) = \frac{2}{\pi}\mathrm{PV}\int_L (\mathbf{u}_a \times g_+) \times_{z_0}^{z} \left(\frac{1}{z - z_0}\right) * \mathrm{d}z + \mathbf{A}.$$

Eq. (18) follows from the above equation by multiplying it by $1/g_+$.

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